CRANBROOK SCHOOL

YEAR 12 MATHEMATICS- Extension 2

Trial HSC examination

Term 3 2003 24-7-03 Time : 3 h / CJL

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Standard Integrals appear at the end of the paper.

Submit your work for Questions 1 and 2 in the same 8 page booklet.
Submit your work for Questions 3 and 4 in the same 8 page booklet.
Submit your work for Questions 5 and 6 in the same 8 page booklet.
Submit your work for Questions 7 and 8 in the same 8 page booklet.

1. (15 marks) Begin a new 8 page booklet for Questions 1 and 2.

a) Sketch the graph of $y = e^{-x}$ **6**
Using this graph and without the use of calculus sketch on separate number planes the following:

(i) $y = -e^{-x}$

(ii) $y = 1 - e^{-x}$

(iii) $y = \dfrac{1}{1-e^{-x}}$

(iv) $y = \left|\dfrac{1}{1-e^{-x}}\right|$

b) On separate number planes sketch the graphs of $y = f(x)$ where:

(i) $f(x) = \sin\left(\cos^{-1} x\right)$ **9**

(ii) $f(x) = \dfrac{x^2}{x-1}$

(iii) $|f(x)| = |x| - 2$

(iv) $f(x) = \dfrac{1}{\sqrt{x^2-1}}$

2. **(15 marks)**

a) If $z=\dfrac{1-i}{3+3\sqrt{3}i}$ **5**

(i) Find the exact value of $|z|$ and $\arg(z)$

(ii) Evaluate z^6

b) Express the square roots of $-2i$ in the form $a+ib$ **2**

c) (i) On the same diagram draw a neat sketch of the loci specified by $|z-3-2i|=2$ and $|z+3|=|z-5|$ **5**

(ii) Hence write down the solutions of z which satisfy simultaneously $|z-3-2i|=2$ and $|z+3|=|z-5|$.

(iii) Use your diagram in (i) to determine the values of k for which the simultaneous equations $|z-3-2i|=2$ and $|z-2i|=k$ have exactly one solution for z.

d) In the Argand diagram the points A, B, C and D represent the complex numbers α,β,γ and δ respectively. **3**

(i) Describe the point which represents $\frac{1}{2}(\alpha+\gamma)$

(ii) Deduce that if $\alpha+\gamma=\beta+\delta$ then ABCD is a parallelogram.

3. **(15 marks) Begin a new 8 page booklet for Questions 3 and 4.**

a) Consider the ellipse $75x^2+100y^2=7500$ **7**

(i) Find the eccentricity

(ii) Find the co-ordinates of the foci

(iii) Find the equations of the directrices

(iv) Find the equation of the normal at the point $(5,7.5)$.

b) Consider the rectangular hyperbola with equation $xy=16$ **8**

A chord, ST, is formed by joining the points $S\left(4s,\dfrac{4}{s}\right)$ and $T\left(4t,\dfrac{4}{t}\right)$

(i) Show that the equation of ST is $x+sty=4(s+t)$

(ii) It is given that ST passes through the point (8,8), show that $2st=s+t-2$.

(iii) Show that the tangents at S and T meet at $R\left(\dfrac{8st}{s+t},\dfrac{8}{s+t}\right)$

(iv) Find the locus of R.

4. **(15 marks)**

a) Let p, q, r be the roots of the equation $x^3 + cx + d = 0$ where $d \neq 0$. Write down the cubic equation in x whose roots are:

(i) p^{-1}, q^{-1}, r^{-1} **4**

(ii) p^2, q^2, r^2

b) (i) Prove that if t is a multiple root of the polynomial equation $g(x) = 0$ then $g(t) = 0$ and $g'(t) = 0$. **5**

(ii) The polynomial equation $x^5 - tx^2 + q = 0$ where t and q are constants, has a multiple root. Show that $108t^5 = 3125q^3$

c) Find a polynomial equation in x of degree 3 such that two zeros are $x = 1$ and $x = -2$ and also $P(-1) = 4$ and $P(2) = 28$. **2**

d) Find the equation of the circle which passes through the points P $(-2,-1)$ and Q $(5,-2)$ and whose centre lies on the line $3x + y + 2 = 0$. **4**

5. **(15 marks) Begin a new 8 page booklet for Questions 5 and 6.**

a) Find (i) $\int \frac{t^2 - 1}{t^3} \, dt$ **4**

(ii) $\int \frac{dx}{\sqrt{6 - x - x^2}}$

b) Evaluate (i) $\int_1^2 \frac{11 - 2t}{(2t-1)(3-t)} \, dt$ **6**

(ii) $\int_1^3 x^2 \ln x \, dx$

c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ where n is anon-negative integer. **5**

(i) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ when $n \geq 2$

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

6. (15 marks)

a) The region between the curve $y = \ln x$, the line $x = e$ and the x axis is rotated about the line $x = e$. By taking slices parallel to the x axis, find the volume generated. **5**

b) The region bounded by $y = x^3$, the x axis and the straight line $x = 2$ is rotated about the straight line $x = 4$. Use the method of cylindrical shells to find the *exact* volume generated. **5**

c) The base of a particular solid is the region bounded by the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$ between its vertex ~~(0,2)~~ (2,0) and its corresponding latus rectum (focal chord perpendicular to major axis). Every cross section of the solid perpendicular to the major axis of the hyperbola is an isosceles right angled triangle with hypotenuse on the base of the solid.

(i) Show that the latus rectum has equation $x = 4$.

5

(ii) Find the volume of the solid.

7. (15 marks) Begin a new 8 page booklet for Questions 7 and 8.

a) Find the following indefinite integrals:

5

(i) $\int \sin 4x \cos 3x \, dx$

(ii) $\int \sin^5 x \cos^4 x \, dx$

b) Find all values of x such that $\sin x = \cos 5x$ for $0 < x < \pi$ **3**

c) Use de Moivre's theorem to express $\cos 6\theta$ as a polynomial in $\cos\theta$.

(Hint: $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$)

Hence: (i) show that the roots of $32x^6 - 48x^4 + 18x^2 - 1 = 0$ are $\pm\dfrac{1}{\sqrt{2}}, \quad \pm\cos\dfrac{\pi}{12}, \quad \pm\cos\dfrac{5\pi}{12}$. **7**

(ii) find the roots of $16x^4 - 16x^2 + 1 = 0$

8. **(15 marks)**

a) Each of the following statements is either true or false. Without evaluating, write TRUE or FALSE for each statement and give brief reasons for your answers. **6**

(i) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \, d\theta = 0$

(ii) $\int_{-1}^{1} e^{-x^2} \, dx = 0$

(iii) $\int_{0}^{\frac{\pi}{2}} \sin^8\theta - \cos^8\theta \, d\theta = 0$

b) It is given that the hyperbola $xy = c^2$ touches (is tangential to) the parabola $y = x - x^2$. **4**

(i) Show this information on a sketch.

(ii) Deduce that the equation $x^3 - x^2 + c^2 = 0$ has a repeated root and hence find the value of c^2.

(iii) Find the co-ordinates of the point where the hyperbola crosses the parabola.

c) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. (let $u = a - x$) **5**

(ii) Consider $f(x) = \dfrac{1}{1+\tan x}$ where $0 \le x \le \dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right) = 0$.

Show that $f(x) + f\left(\dfrac{\pi}{2} - x\right) = 1$. (note: $\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$.

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \dfrac{1}{1+\tan x} dx$

END OF EXAMINATION

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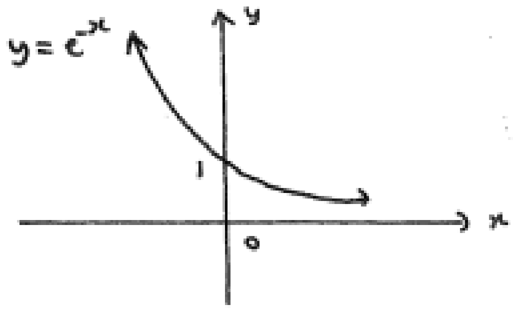
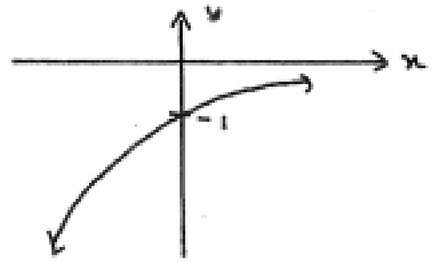
Question 1

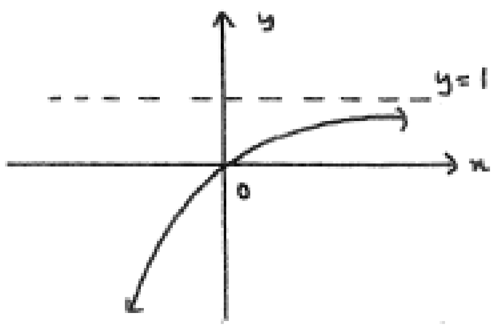
a)



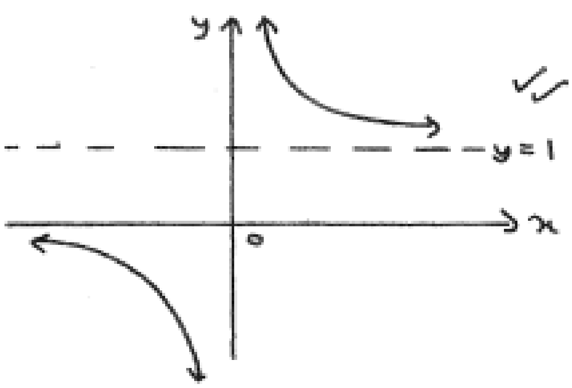
(i)



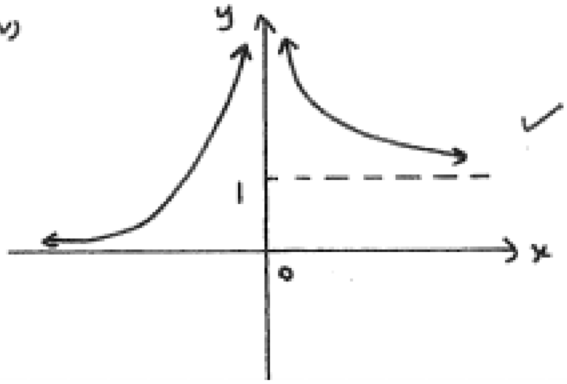
(ii)



(iii)

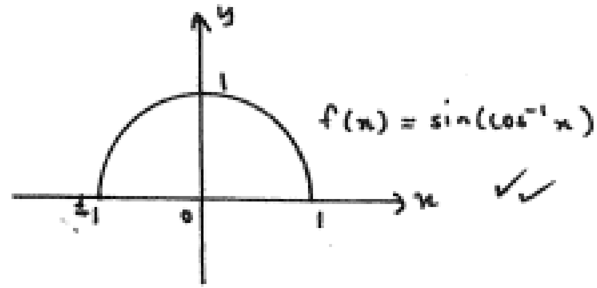


(iv)



b)

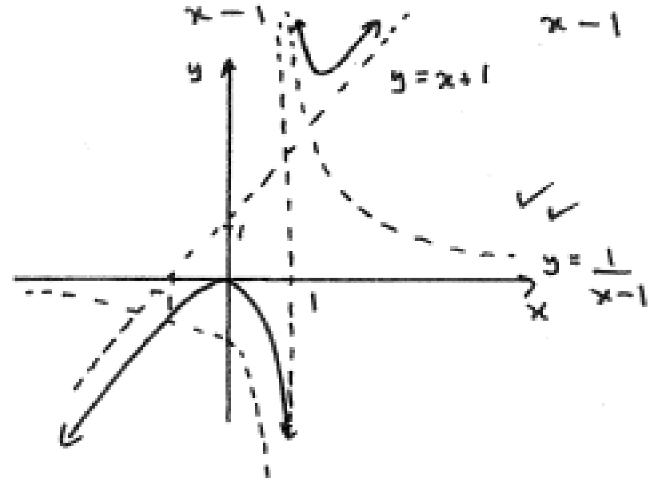
(i)



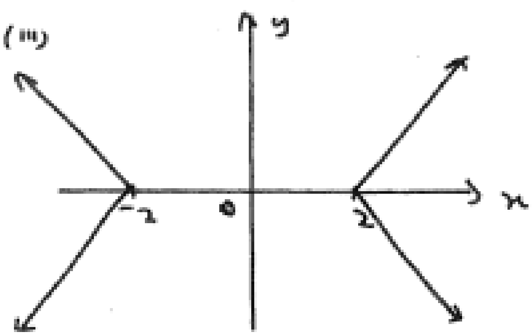
(ii)

$$\begin{array}{r}
x + 1 \\
x - 1 \overline{)\, x^2 + 0x + 0} \\
\underline{x^2 - x} \\
x + 0 \\
\underline{x - 1} \\
1
\end{array}$$

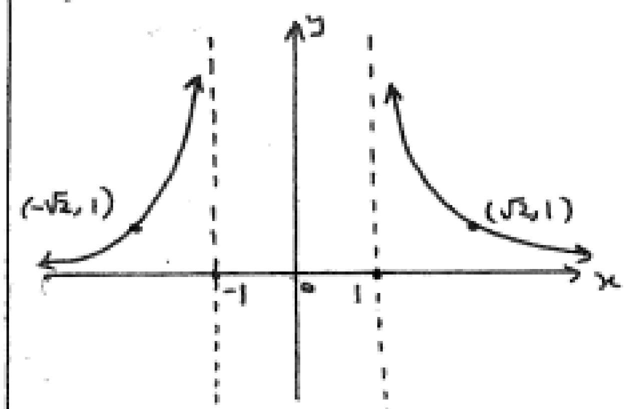
$\therefore f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$



(iii)



(iv)



) $z = \dfrac{1-i}{3+3\sqrt{3}i}$

et $z_1 = 1 - i$ $\quad |z_1| = \sqrt{2}$

$\arg(z_1) = -\dfrac{\pi}{4}$ ✓

et $z_2 = 3 + 3\sqrt{3}i$ $\quad |z_2| = 6$

$\arg(z_2) = \dfrac{\pi}{3}$

$\therefore |z| = \dfrac{\sqrt{2}}{6}$ ✓

$\arg(z) = -\dfrac{\pi}{4} - \dfrac{\pi}{3} = -\dfrac{7\pi}{12}$ ✓

ii) $z = \dfrac{\sqrt{2}}{6} \operatorname{cis}\left(\dfrac{-7\pi}{12}\right)$

$\therefore z^6 = \left(\dfrac{\sqrt{2}}{6}\right)^6 \operatorname{cis}\left(6 \times \dfrac{-7\pi}{12}\right)$ ✓

$= \dfrac{8}{46656}\left[\cos\left(\dfrac{-7\pi}{2}\right) + i\sin\left(\dfrac{-7\pi}{2}\right)\right]$

$= \dfrac{i}{5832}$ ✓

i) let $\sqrt{-2i} = a + ib$

$\therefore -2i = a^2 - b^2 + 2abi$

$ab = -1$ $\qquad a^2 - b^2 = 0$

$b = -\dfrac{1}{a}$ $\qquad a^2 - \dfrac{1}{a^2} = 0$

$a^4 - 1 = 0$

$(a^2-1)(a^2+1) = 0$

$a = \pm 1$

$\therefore b = \mp 1$

$\therefore \sqrt{-2i} = \pm(1 - i)$

OR

let $z^2 = -2i$

$= 2\operatorname{cis}\left(-\dfrac{\pi}{2}\right)$ ✓

$\therefore z = \pm\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$z = \pm(1 - i)$ ✓

c) (i) $|z - 3 - 2i| = 2$

$|x - 3 + i(y-2)| = 2$

circle centre $(3, 2)$

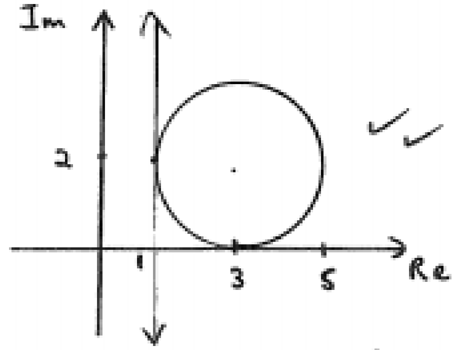
$|x + 3 + iy| = |x - 5 + iy|$

$(x+3)^2 + y^2 = (x-5)^2 + y^2$

$x^2 + 6x + 9 + y^2 = x^2 - 10x + 25 + y^2$

$16x = 16$

$x = 1$



✓✓

(ii) $z = 1 + 2i$ (point of int $(1,2)$) ✓

(iii) $|z - 2i| = k$

$|x + i(y-2)| = k$

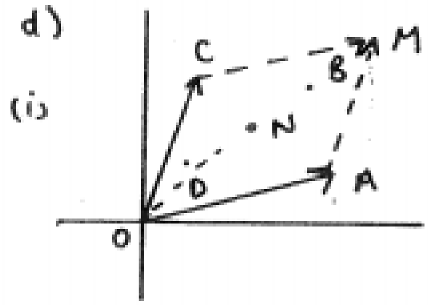
circle centre $(0, 2)$

radius k

$\therefore k = 1$ or $k = 5$ ✓✓

d)

(i)



A is α

C is γ

M is $\alpha + \gamma$ where OAMC is a parallelogram ✓

N is $\frac{1}{2}(\alpha + \gamma)$ where N is midpt of OM and AC ✓

(ii) $\alpha + \gamma = \beta + \delta$ given

$\therefore \frac{1}{2}(\alpha + \gamma) = \frac{1}{2}(\beta + \delta)$ ✓

$\therefore$ midpt of AC = midpt of BD

$\therefore$ ABCD is a parallelogram

i) $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$

$a = 10 \quad b = \sqrt{75} = 5\sqrt{3}$

; $b^2 = a^2(1-e^2)$ ✓

$75 = 100(1-e^2)$

$\frac{3}{4} = 1 - e^2$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$ ✓

) foci $(\pm ae, 0)$

$(\pm 5, 0)$ ✓

iii) directrices

$x = \pm \frac{a}{e}$

$x = \pm 20$ ✓

iv) $\dfrac{2x}{100} + \dfrac{2y}{75}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{3x}{4y}$ ✓

at $(5, 7.5)$ $\dfrac{dy}{dx} = \dfrac{-1}{2}$

$\therefore m_N = 2$ ✓

eqn. of normal is

$y - 7.5 = 2(x-5)$

$y - 7.5 = 2x - 10$

$y = 2x - 2.5$ ✓

b) (i) gradient of ST is

$$\frac{\frac{4}{s} - \frac{4}{t}}{4s - 4t} = \frac{\frac{4(t-s)}{st}}{4(s-t)}$$

$\therefore m_{ST} = -\dfrac{1}{st}$ ✓

eqn. of ST

$y - \frac{4}{s} = -\frac{1}{st}(x - 4s)$

$sty - 4t = -x + 4s$

$\therefore x + sty = 4(s+t)$ ✓

(ii) sub $(8, 8)$

$8 + 8st = 4(s+t)$ ✓

(iii) $y = 16x^{-1}$

$y' = -\dfrac{16}{x^2}$ at $x = 4s$ ✓

$y' = \dfrac{-16}{16s^2} = -\dfrac{1}{s^2}$

eqn. of tangent at S

$y - \frac{4}{s} = -\frac{1}{s^2}(x - 4s)$

$s^2y - 4s = -x + 4s$

$x + s^2y = 8s$ — ① ✓

$\therefore$ eqn of tangent at T

$x + t^2y = 8t$ — ②

① − ② $y(s^2 - t^2) = 8(s-t)$

$y = \dfrac{8(s-t)}{(s-t)(s+t)}$

$y = \dfrac{8}{s+t}$ ✓

$\therefore x + \dfrac{s^2 8}{s+t} = 8s$

$x = 8s - \dfrac{8s^2}{s+t}$

$x = \dfrac{8s^2 + 8st - 8s^2}{s+t}$

$x = \dfrac{8st}{s+t}$ ✓

$\therefore R$ is $\left(\dfrac{8st}{s+t}, \dfrac{8}{s+t}\right)$

(iv) Now $y = \dfrac{8}{s+t}$ $\therefore s + t = \dfrac{8}{y}$

$\therefore x = \dfrac{8st}{8/y}$

$x = yst$

$\frac{x}{y} = st$

Now $2 + 2st = s + t$

$2 + \frac{2x}{y} = \frac{8}{y}$

$2y + 2x = 8$

$\therefore x + y = 4$ ✓

(ii) roots are $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$

sub $x = \frac{1}{x}$

$\therefore \left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right) + d = 0$

$1 + cx^2 + dx^3 = 0$

(iii) roots are p^2, q^2, r^2

sub $x = \sqrt{x}$

$\therefore (\sqrt{x})^3 + c\sqrt{x} + d = 0$

$x\sqrt{x} + c\sqrt{x} + d = 0$

$\sqrt{x}\,(x+c) = -d$

$x\,(x+c)^2 = d^2$

$x(x^2 + 2cx + c^2) = d^2$

$x^3 + 2cx^2 + c^2x - d^2 = 0$

) (i) let $g(x) = (x-t)^r . Q(x)$, $Q(x) \neq 0$.

$g'(x) = (x-t)^r . Q'(x) + Q(x) . r(x-t)^{r-1}$

$= (x-t)^{r-1}[(x-t)Q'(x) + rQ(x)]$

$\therefore g'(t) = (t-t)^{r-1}[(t-t)Q'(t) + rQ(t)]$

$= 0$

If t is a multiple root of $g(x) = 0$,

, $g(t) = g'(t) = 0$.

(ii) $f(x) = x^5 - tx^2 + q = 0$

$f'(x) = 5x^4 - 2xt = 0$

$\therefore 5x^4 = 2xt$

$t = \frac{5x^3}{2}$

$\therefore x^5 - \frac{5x^3}{2}.x^2 + q = 0$

$x^5 - \frac{5x^5}{2} = -q$

$\therefore q = \frac{3x^5}{2}$

Now $t^5 = \left(\frac{5x^3}{2}\right)^5 = \frac{3125x^{15}}{32}$

and $q^3 = \left(\frac{3x^5}{2}\right)^3 = \frac{27x^{15}}{8}$

$\therefore \frac{t^5}{q^3} = \frac{3125}{32} \div \frac{27}{8}$

$= \frac{3125}{108}$

$\therefore 108t^5 = 3125q^3$

Now $4 = a(-2)(1)(-1-\alpha)$

$4 = 2a(1+\alpha)$

$a = \frac{2}{1+\alpha}$

And $28 = a(1)(4)(2-\alpha)$

$28 = 4a(2-\alpha)$

$a = \frac{7}{2-\alpha}$

$\therefore \frac{2}{1+\alpha} = \frac{7}{2-\alpha}$

$4 - 2\alpha = 7 + 7\alpha$

$-3 = 9\alpha$

$\alpha = -\frac{1}{3}$

$\therefore a = \frac{2}{2/3} = 3$

$\therefore P(x) = 3(x-1)(x+2)(x+\frac{1}{3})$

$= (x-1)(x+2)(3x+1)$

d) Perp. bisector of PQ is

$m = \frac{-1+2}{-2-5} = \frac{1}{-7}$

$\therefore m_\perp = 7$

$\therefore$ M.P. $= \left(\frac{-2+5}{2}, \frac{-1-2}{2}\right) = \left(\frac{3}{2}, -\frac{3}{2}\right)$

$\therefore y + \frac{3}{2} = 7(x - \frac{3}{2})$

$y + \frac{3}{2} = 7x - \frac{21}{2}$

$7x - y = 12$ +

$3x + y = -2$

$10x = 10$

$x = 1 \quad y = -5$

centre of circle is $(1, -5)$

radius $= \sqrt{(1+2)^2 + (-5+1)^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= 5$

$\therefore$ circle has equation

$(x-1)^2 + (y+5)^2 = 25$

(i) $\int \frac{t^2-1}{t^3}\,dt = \int \frac{1}{t} - t^{-3}\,dt$

$= \ln|t| + \frac{1}{2t^2} + c$ ✓✓

(ii) $\int \frac{dx}{\sqrt{6-x-x^2}} = \int \frac{dx}{\sqrt{\frac{25}{4}-(x+\frac{1}{2})^2}}$

$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{5}{2}}\right) + c = \sin^{-1}\left(\frac{2x+1}{5}\right) + c$ ✓✓

(i) let $\frac{11-2t}{(2t-1)(3-t)} = \frac{a}{2t-1} + \frac{b}{3-t}$

$11-2t = a(3-t) + b(2t-1)$

$t = 3 \quad \therefore 5 = b(5)$

$b = 1$

$t = \frac{1}{2} \quad \therefore 10 = a(2\frac{1}{2})$

$a = 4$

$\int_1^2 \frac{11-2t}{(2t-1)(3-t)}\,dt$

$= \int_1^2 \frac{4}{2t-1} + \frac{1}{3-t}\,dt$ ✓

$= \left[2\ln|2t-1| - \ln|3-t|\right]_1^2$ ✓

$= (2\ln 3 - \ln 1) - (2\ln 1 - \ln 2)$

$= 2\ln 3 + \ln 2$

$= \ln 18$ ✓

(ii) $u = \ln x \qquad dv = x^2\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{3}x^3$ ✓

$\int_1^3 x^2 \ln x\,dx$

$= \left[\frac{1}{3}x^3 \ln x\right]_1^3 - \int_1^3 \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$ ✓

$= 9\ln 3 - \frac{1}{3}\ln 1 - \frac{1}{3}\int_1^3 x^2\,dx$

$= 9\ln 3 - \left[\frac{1}{9}x^3\right]_1^3$

$= 9\ln 3 - \left(3 - \frac{1}{9}\right)$

$= 9\ln 3 - 2\,^8\!/_9$ ✓

(i) $= \int_0^{\frac{\pi}{2}} \sin^{n-1}x \cdot \sin x\,dx$

$u = \sin^{n-1}x$

$du = (n-1)\sin^{n-2}x\cos x\,dx$

$dv = \sin x\,dx$ ✓

$v = -\cos x$

$\therefore I_n = \left[-\cos x \sin^{n-1}x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x\,d$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x(1-\sin^2 x)\,d$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x - \sin^n x\,dx$ ✓

$I_n = (n-1)I_{n-2} - (n-1)I_n$

$I_n = (n-1)I_{n-2} - nI_n + I_n$

$nI_n = (n-1)I_{n-2}$

$I_n = \frac{n-1}{n}I_{n-2}$ ✓

(ii) $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = I_5$ ✓

$I_5 = \frac{4}{5}I_3$

$I_3 = \frac{2}{3}I_1$

$I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx$

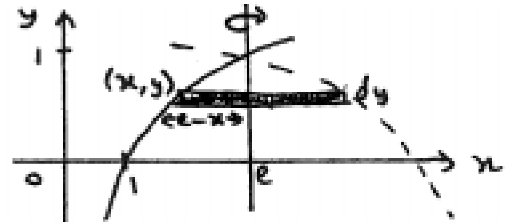
$= \left[-\cos x\right]_0^{\pi/2}$

$= (0) - (-1)$

$= 1$

$\therefore I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$= \frac{8}{15}$ ✓



Vol. of one slice is

$$\delta V = \pi (e-x)^2 \delta y$$

Vol of all slices is

$$\delta V = \lim_{\delta y \to 0} \sum_{0}^{1} \pi (e-x)^2 \delta y$$

$$\therefore V = \pi \int_0^1 e^2 - 2ex + x^2 \, dy$$

Now $y = \ln x$

$\therefore x = e^y$

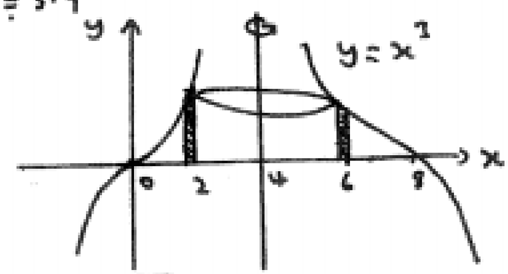
$$V = \pi \int_0^1 e^2 - 2e.e^y + e^{2y} \, dy$$

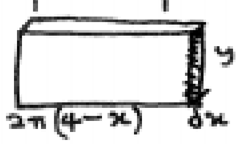
$$= \pi \left[e^2 y - 2e.e^y + \tfrac{1}{2} e^{2y} \right]_0^1$$

$$= \pi \left[\left(e^2 - 2e.e + \tfrac{1}{2} e^2 \right) - \left(0 - 2e + \tfrac{1}{2} \right) \right]$$

$$= \pi \left[2e - \tfrac{1}{2} e^2 - \tfrac{1}{2} \right] \text{ units}^3$$

$\doteqdot 3.9$





Vol. of one shell is

$$\delta V = 2\pi (4-x) y \, \delta x$$

Vol of all shell is

$$\delta V = \lim_{\delta x \to 0} 2\pi \sum_{0}^{2} (4-x) y \, \delta x$$

$$\therefore V = 2\pi \int_0^2 4y - xy \, dx$$

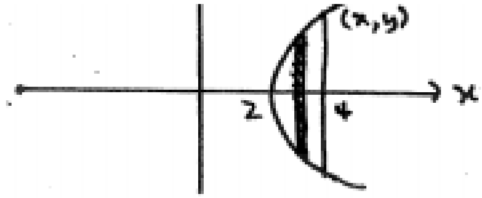
$$= 2\pi \int_0^2 4x^3 - x^4 \, dx$$

$$= 2\pi \left[x^4 - \tfrac{1}{5} x^5 \right]_0^2$$

$$= 2\pi \left(16 - \tfrac{32}{5} \right)$$

$$= 96\pi \text{ units}^3$$

(i)



$a = \sqrt{4} = 2$

$b = \sqrt{12} = 2\sqrt{3}$

$b^2 = a^2(e^2 - 1)$

$12 = 4(e^2 - 1)$

$3 = e^2 - 1$

$e^2 = 4$

$e = 2$

foci $(\pm ae, 0)$

$(\pm 4, 0)$

$\therefore$ latus rectum is $x = 4$

(ii)

Vol of one slice is

$$\delta V = \tfrac{1}{2} \cdot 2y \cdot y \cdot \delta x$$

Vol of all slices is

$$\delta V = \lim_{\delta x \to 0} \int_2^4 y^2 \, \delta x$$

$$\therefore V = \int_2^4 y^2 \, dx$$

$$= \int_2^4 3x^2 - 12 \, dx$$

$$= \left[x^3 - 12x \right]_2^4$$

$$= (64 - 48) - (8 - 24)$$

$$= 32 \text{ units}^3$$

(i) $\int \sin 4x \cos 3x \, dx$

$= \frac{1}{2}\int \sin x + \sin 7x \, dx$ ✓

$= \frac{1}{2}\left(-\cos x - \frac{1}{7}\cos 7x\right) + c$ ✓

(ii) $\int \sin^5 x \cos^4 x \, dx$

$= \int \sin^4 x \cos^4 x \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$ ✓

$= \int (1 - 2\cos^2 x + \cos^4 x)\cos^4 x \sin x \, dx$

$= \int \cos^4 x \sin x - 2\int \cos^6 x \sin x \, dx$

$+ \int \cos^8 x \sin x \, dx$ ✓

$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + c$ ✓

) $\sin x = \cos 5x$

$\cos\left(\frac{\pi}{2} - x\right) = \cos 5x$ ✓

$\therefore 5x = \frac{\pi}{2} - x$ ✓

$5x = \frac{\pi}{2} - x$

$6x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{9\pi}{4}, \frac{21\pi}{4}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}$

✓

$z^6 = \cos 6\theta + i\sin 6\theta$

and $z^6 = (\cos\theta + i\sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta\, i\sin\theta + 15\cos^4\theta\, i^2\sin^2\theta$

$+ 20\cos^3\theta\, i^3\sin^3\theta + 15\cos^2\theta\, i^4\sin^4\theta$

$+ 6\cos\theta\, i^5\sin^5\theta + i^6\sin^6\theta$

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta$ ✓

$- 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta$

$+ 6i\cos\theta\sin^5\theta - \sin^6\theta$

equate reals

$\therefore \cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta$

$+ 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta)$ ✓

$+ 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta$

$+ 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$-(1 - 3\cos^2\theta + 3\cos^4\theta \quad \cos^6\theta)$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta$

$- 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta$

$- 3\cos^4\theta + \cos^6\theta$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta -$ ✓

(i) let $x = \cos\theta$

$\therefore 32x^6 - 48x^4 + 18x^2 - 1 = 0$

ie $\cos 6\theta = 0$ ✓

$\therefore 6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

$\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}$

$\therefore x = \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}, \pm\cos\frac{\pi}{4}$

$x = \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}, \pm\frac{1}{\sqrt{2}}$ ✓

(ii) Now $\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right)$ are factor of $P(x)$ ie $(2x^2 - 1)$ is a fact ✓

And

$32x^6 - 48x^4 + 18x^2 - 1 = (2x^2 - 1)(16x^4 - 16x^2 + 1)$ by inspection

$\therefore$ roots of $16x^4 - 16x^2 + 1 = 0$

$\ldots \pm\cos\frac{\pi}{\ldots} \quad \pm\cos\frac{5\pi}{\ldots}$ ✓

)(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \, d\theta = 0$ TRUE

inθ is odd $\therefore \sin^7\theta$ is odd

tegral of an odd function between

ymmetrical limits is zero

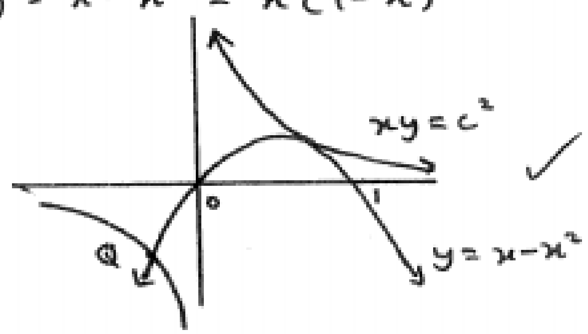
ii) $\int_{-1}^{1} e^{-x^2} dx = 0$ FALSE

$e^{-x^2} > 0$ for all x $\therefore \int_{-1}^{1} e^{-x^2} dx > 0$

i) $\int_0^{\frac{\pi}{2}} \sin^8\theta - \cos^8\theta \, d\theta = 0$ TRUE

$\cos\theta = \sin(\frac{\pi}{2} - \theta)$

) $y = x - x^2 = x(1-x)$



ii) $x(x - x^2) = c^2$

$x^2 - x^3 = c^2$

$\therefore x^3 - x^2 + c^2 = 0$

Consider $P(x) = x^3 - x^2 + c^2$

$P'(x) = 3x^2 - 2x$

$\therefore 3x^2 - 2x = 0$

$x(3x - 2) = 0$

$x = 0 \quad x = \frac{2}{3}$ must be a repeated root

$P(\frac{2}{3}) = \frac{8}{27} - \frac{4}{9} + c^2 = 0$

$c^2 = \frac{4}{27}$

iii) $P(x) = (x - \frac{2}{3})^2 (x + \frac{1}{3})$

$\therefore$ Q is $(-\frac{1}{3}, -\frac{4}{9})$

(i) $x = a \Rightarrow u = 0$

$du = -dx$

$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(u) \cdot -du$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

(ii) $f(x) = \frac{1}{1 + \tan x}$

$f(\frac{\pi}{2} - x) = \frac{1}{1 + \tan(\frac{\pi}{2} - x)}$

$= \frac{1}{1 + \cot x}$

$\therefore f(x) + f(\frac{\pi}{2} - x)$

$= \frac{1}{1+\tan x} + \frac{1}{1 + \cot x}$

$= \frac{1}{1+\tan x} + \frac{1}{1 + \frac{1}{\tan x}}$

$= \frac{1}{1+\tan x} + \frac{1}{\frac{\tan x + 1}{\tan x}}$

$= \frac{1}{1+\tan x} + \frac{\tan x}{\tan x + 1}$

$= 1$

(iii) Now

$\int_0^{\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} f(\frac{\pi}{2} - x)\,dx$

$= \int_0^{\frac{\pi}{2}} 1 . dx$

$\therefore \quad I + I = [x]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx = \frac{\pi}{4}$